

# The Milnor-Thurston determinant and the Ruelle transfer operator.

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## Abstract

The topological entropy  $h_{\text{top}}$  of a continuous piecewise monotone interval map measures the exponential growth in the number of monotonicity intervals for iterates of the map. Milnor and Thurston showed that  $\exp(-h_{\text{top}})$  is the smallest zero of an analytic function, now coined the Milnor-Thurston determinant, that keeps track of relative positions of forward orbits of critical points. On the other hand  $\exp(h_{\text{top}})$  equals the spectral radius of a Ruelle transfer operator  $L$ , associated with the map. Iterates of  $L$  keep track of inverse orbits of the map. For no obvious reason, a Fredholm determinant for the transfer operator has not only the same leading zero as the M-T determinant but all peripheral (those lying in the unit disk) zeros are the same.

The purpose of this note is to show that on a suitable function space, the dual of the Ruelle transfer operator has a regularized determinant, identical to the Milnor-Thurston determinant, hereby providing a natural explanation for the above puzzle.

## 1 Introduction

In the 70s, Milnor and Thurston came up with an intriguing way of computing the topological entropy  $h_{\text{top}}$  for a continuous piecewise monotone interval map  $f$ . They invented a "kneading-matrix",  $M(t)$ , a finite matrix-valued powerseries in an auxiliary variable  $t$ . The matrix keeps track of relative positions of forward orbits of critical points relative to the critical points themselves. The "Milnor-Thurston" determinant,  $d_{\text{MT}}(t) = \det M(t)$ , defines an analytic function in the unit disk and they showed that if  $h_{\text{top}} > 0$  then  $t_* = \exp(-h_{\text{top}}) \in (0, 1)$  is a zero of  $d_{\text{MT}}(t)$ . The zero is 'extremal' in the sense that it is the smallest in absolute value. Their result is computationally useful, as the effort in a direct computation of the number of monotonicity intervals of  $f^n$  grows like  $e^{nh_{\text{top}}}$  whereas computing  $M(t)$  to a similar accuracy only grows linearly with  $n$ .

The above-mentioned extremal property was shown indirectly by proving that  $d_{\text{MT}}(t)$  is identical to a certain Lefschetz zeta-function, generated from the periodic orbits for  $f$ . The identity is shown in one (simple) case and using continuous deformations of the map it is shown that the identity persists under such deformations. This part is rather puzzling, as no arguments indicate why one would expect such a relation.

Also in the 70s, Ruelle used transfer operators to define (generalized) Fredholm determinants and zeta-functions. In our context, an operator  $L$  would act upon functions of bounded variation. Iterates of  $L$  keep track of inverse orbits of points and its spectral radius equals precisely  $1/t_* = \exp(h_{\text{top}})$ . Baladi-Keller showed [BK], that an associated determinant  $d_{\text{R}}(t) = \det(1 - tL)$  is analytic in the unit disk and (at least for expanding maps) it is easy to see that the zeros within the unit disk are the same as for the Lefschetz zeta function whence also for  $d_{\text{MT}}(t)$ . This suggests a deep relation between the M-T determinant and the Ruelle-determinant of  $L$ , whence with the operator  $L$  itself.

A step towards clarifying this relationship was suggested through a study of the dual operator by Baladi and Ruelle [BR], but the functional setup makes explicit computations hard and requires operator-weights to be globally continuous. Gouëzel [G], elaborating the functional setup further, still requires weights to be continuous at periodic points. Both of these results exclude partially the case studied by Milnor and Thurston.

The goal here is to show that the determinant of Milnor-Thurston may be obtained by restricting  $L$  to a suitable small subspace  $X$  of Bounded Variation functions, and calculate an elementary regularized determinant of the dual operator  $L'$ . We first exhibit a natural isometry between the dual space  $X'$  and a space  $\hat{X}$  of uniformly bounded functions of "point-germs". This gives an explicit representation for the dual operator,  $\hat{L} = S - PS$ , in which  $S$  has spectral radius 1 and  $P$  is of finite rank. When  $|\lambda| > 1$  (we call such a value a 'peripheral' value) then  $\lambda - S$  is invertible through a von Neumann series. The elementary formula,

$$\lambda - (S - PS) = (1 + PS(\lambda - S)^{-1})(\lambda - S),$$

shows that  $\lambda$  is a spectral value iff the operator  $(1 + PS(\lambda - S)^{-1})$  is non-invertible. As  $P$  is of finite rank, this happens iff the following finite dimensional determinant vanishes:

$$\det(1 + PS(\lambda - S)^{-1}) := \det_{\text{im } P} \left( P \sum_{k \geq 0} \lambda^{-k} S^k \right). \quad (1)$$

Finally, setting  $\lambda = 1/t$  this turns out to be the Milnor-Thurston determinant. An advantage of the present approach is that we may exploit positivity of the Ruelle operator (see Theorem 3.3 below) to bypass the use of the Lefschetz zeta-function in the proof of the extremal property of the zero.

## 2 Step functions of bounded variation.

Fix  $-\infty \leq a < b \leq +\infty$ . The variation of a function  $\phi : (a, b) \rightarrow \mathbb{C}$  is:

$$\text{var } \phi = \sup \left\{ \sum_i |\phi(x_i) - \phi(x_{i+1})| : a < x_1 < \dots < x_N < b \right\}. \quad (2)$$

We say that  $\phi$  is of bounded variation (BV) iff  $\text{var } \phi < +\infty$ . For a BV-function right and left limits always exist and following usual conventions we write  $\phi(x^+) = \lim_{t \rightarrow 0^+} \phi(x + t)$  and  $\phi(x^-) = \lim_{t \rightarrow 0^+} \phi(x - t)$  defined for  $x \in [a, b)$  and  $x \in (a, b]$ , respectively. We assume no a priori relation between the values of  $\phi$  at  $x^-$ ,  $x$  and  $x^+$  (see also Remark 3.2 below). We define the 'boundary' value of  $\phi$  to be

$$\partial\phi = \phi(a^+) + \phi(b^-). \quad (3)$$

In the following we write  $\hat{x} = x^\epsilon$  for either  $x^+$  or  $x^-$  and call it a point germ with base point  $x$  and direction  $\epsilon = +1$  or  $\epsilon = -1$ . We order point germs intertwined with base points by declaring that when  $x < y$  then  $x < x^+ < y^- < y$ . Any real segment may then be specified through two point-germs by setting:  $\langle \hat{u}, \hat{v} \rangle = \{x \in (a, b) : \hat{u} < x < \hat{v}\}$ . So e.g.  $\langle u^-, v^- \rangle = [u, v)$  (for  $u < v$ ),  $\langle u^-, u^+ \rangle = \{u\}$  and  $\langle \hat{u}, \hat{u} \rangle = \emptyset$ .

**Definition 2.1.** A map  $\phi : (a, b) \rightarrow \mathbb{C}$  is said to be a simple step function if there is a finite set  $C = C_\phi \subset (a, b)$  so that  $\phi$  is constant on each connected component of  $(a, b) \setminus C$ . We write  $\mathcal{S} = \mathcal{S}(a, b)$  for the space of simple step functions on  $(a, b)$ .

When  $a < x_0 < x < x_N < b$  then  $2|\phi(x)| \leq \text{var } \phi + |\phi(x_0) + \phi(x_N)|$ . Letting  $x_0 \rightarrow a^+$  and  $x_N \rightarrow b^-$  yields  $\sup |\phi| \leq \frac{1}{2}(\text{var } \phi + |\partial\phi|)$ . One easily obtains :

**Proposition 2.1.** The quantity  $\|\phi\|_{\text{BV}} = \text{var } \phi + |\partial\phi|$  defines a norm on  $\mathcal{S}$ . We let  $X$  denote the completion of  $\mathcal{S}$  under this norm and call  $X = \text{SBV}(a, b)$  the space of Step functions of Bounded Variation on  $(a, b)$ . One has  $\sup |\phi| \leq \frac{1}{2}\|\phi\|_{\text{BV}}$  for  $\phi \in X$ .

We define for every  $a^+ \leq \hat{u} \leq b^-$  the following "base" step-function (which is everywhere non-zero):

$$\sigma_{\hat{u}}(x) = \frac{1}{2} \times \begin{cases} +1 & , \hat{u} < x < b \\ -1 & , a < x < \hat{u} \end{cases} \quad (4)$$

In particular,  $\sigma_{a^+}(x) = -\sigma_{b^-}(x) = \frac{1}{2}$  for all  $a < x < b$ . In terms of characteristic functions one has:  $\sigma_{\hat{u}}(x) = \frac{1}{2} \left( -\chi_{\langle a^+, \hat{u} \rangle}(x) + \chi_{\langle \hat{u}, b^- \rangle}(x) \right)$ .

Let  $X'$  be the Banach dual of  $X$ , i.e. the space of linear maps  $\ell : X \rightarrow \mathbb{C}$  for which  $\langle \ell, \phi \rangle \leq C\|\phi\|_{\text{BV}}$ ,  $\forall \phi \in X$  and some constant  $C < +\infty$ . The norm  $\|\ell\|_{X'}$  of  $\ell$  is defined as the smallest such constant. Acting with  $\ell$  upon a base step function  $\sigma_{\hat{u}}$  we obtain

a representation of  $\ell$  as a function on the set of point germs  $\widehat{I} = [a^+, b^-]$ . We call this the  $\sigma$ -transform of  $h' \in X'$  and write

$$\widehat{\ell}(\widehat{u}) := \langle \ell, \sigma_{\widehat{u}} \rangle, \quad \widehat{u} \in \widehat{I}. \quad (5)$$

As is readily seen,  $\sigma_{\widehat{u}}$  has norm one, so  $\|\ell\|_\infty \leq \|\ell\|_{X'}$ . Thus,  $\ell$  is an element of the space  $\widehat{X} = B([a^+, b^-])$ , the bounded functions on  $\widehat{I}$  equipped with the uniform norm  $\|\cdot\|_\infty$ . Since  $\langle \ell, \sigma_{b^-} \rangle = -\langle \ell, \sigma_{a^+} \rangle$ , we have  $\ell(a^+) + \ell(b^-) = 0$  so  $\ell$  in fact belongs to the closed subspace  $\widehat{X}_0 = \{H \in \widehat{X} : H(a^+) + H(b^-) = 0\}$ . But more is true:

**Proposition 2.2.** *The  $\sigma$ -transform  $\ell \in X' \mapsto \widehat{\ell} \in \widehat{X}_0$  is a Banach space isometry.*

Proof: The collection  $\{\sigma_{\widehat{u}} : a^+ \leq \widehat{u} < b^-\}$  forms a base for  $\mathcal{S}$ , the space of simple step functions. In fact, given any simple step function  $\phi$  there is a finite number of point germs  $a^+ = \widehat{u}_0 < \dots < \widehat{u}_N < b^-$  so that:

$$\phi = w_0 \frac{1}{2} \mathbf{1} + \sum_{i=1}^N w_i \sigma_{\widehat{u}_i} = \sum_{i=0}^N w_i \sigma_{\widehat{u}_i}. \quad (6)$$

We omit a term involving  $b^-$  since  $-\sigma_{b^-} = \sigma_{a^+}$  and the latter is already included in the sum. The decomposition is unique if every  $w_i \neq 0$ ,  $i > 0$ . In terms of this decomposition:

$$\text{var } \phi = \sum_{i=1}^N |w_i| \quad \text{and} \quad \partial \phi = w_0. \quad (7)$$

To see this, note that the variation is bounded by the right hand sum and as is readily checked, the supremum in the definition (2) may be realized by choosing a mesh that verifies:  $a < x_1 < \widehat{u}_1 < x_2 < \widehat{u}_2 < \dots < \widehat{u}_N < x_{N+1} < b$ . The norm of  $\phi$  is given by:

$$\|\phi\|_{\text{BV}} = \sum_{i=0}^N |w_i|. \quad (8)$$

Given a simple step function  $\phi \in \mathcal{S}$  it may be decomposed as in (6) and using the expression (8) for the norm we see that:

$$\langle \ell, \phi \rangle = \sum_{i=0}^N w_i \ell(\widehat{u}_i) \quad (9)$$

$$|\langle \ell, \phi \rangle| = \sum_{i=0}^N |w_i| |\ell(\widehat{u}_i)| \leq \sum_{i=0}^N |w_i| \|\widehat{\ell}\|_\infty = \|\phi\|_{\text{BV}} \|\widehat{\ell}\|_\infty. \quad (10)$$

Now, this extends by continuity to every  $\phi$  in  $X$ , the completion of  $\mathcal{S}$ . Conversely, any function  $H \in \widehat{X}_0$  gives rise to a linear functional on  $\mathcal{S}$  of the same norm, by setting  $\langle \ell, \phi \rangle := \sum_{i=0}^N w_i H(\widehat{u}_i)$  and extending by continuity  $\ell$  becomes a linear functional  $\ell \in X'$  of the same norm.  $\square$

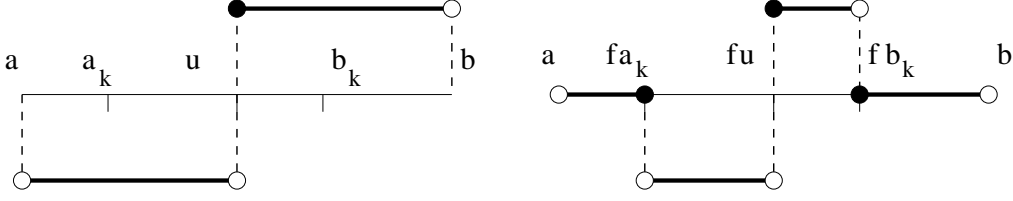


Figure 1: Graph of  $\sigma_{u-}$  and  $L_k \sigma_{u-}$  for  $a_k < u < b_k$  and with  $s_k = +1$ .

**Remarks 2.3.** The space  $\text{SBV}(a, b)$  is isomorphic to  $\mathbb{C} \oplus \ell^1((a, b))$  (summable functions on the uncountable set of point germs + the boundary value). Not surprisingly,  $X'$  is isomorphic to  $\mathbb{C} \oplus \ell^\infty((a, b))$ .

**Proposition 2.4.** Given a bounded linear operator  $A \in L(X)$  we obtain a representation  $\widehat{A} \in L(\widehat{X}_0)$  through:

$$\left(\widehat{A}\widehat{\ell}\right)(\widehat{u}) := \langle A'\ell, \sigma_{\widehat{u}} \rangle = \langle \ell, A\sigma_{\widehat{u}} \rangle, \quad \widehat{u} \in [a^+, b^-]. \quad (11)$$

The operators  $A$  and  $A'$  (whence  $\widehat{A}$ ) have the same spectrum [Kato, III, Thm 6.22].

$\lambda$  is an isolated eigenvalue of finite algebraic multiplicity for  $A$  iff it is for  $A'$  (whence for  $\widehat{A}$ ). In this case, algebraic (and geometric) multiplicities are the same for  $A$ ,  $A'$  and  $\widehat{A}$ . [Kato, III, Remark 6.23].

In the following we shall study a family of operators where the representation of the dual operator takes a particular simple form.

### 3 A Ruelle transfer operator

Let  $\emptyset \neq I_k = (a_k, b_k) \subset (a, b)$  (below,  $k$  will serve as an index) and let  $f_k : I_k \rightarrow (a, b)$  be a continuous, strictly monotone map. We write  $s_k = s(f_k) \in \{\pm 1\}$  for the sign of monotonicity. Also let  $g_k \in \mathbb{C}$  (the topological entropy corresponds to the case  $g_k \equiv 1$ ). Given a triple  $(I_k, f_k, g_k)$  we associate a bounded linear operator,  $L_k$  acting upon  $\phi \in X = \text{SBV}(a, b)$  and defined as follows:

$$L_k \phi(y) = g_k \phi \circ f_k^{-1}(y) \chi_{f_k I_k}(y), \quad y \in (a, b). \quad (12)$$

Acting with  $L_k$  e.g. upon a characteristic function  $\chi_J$  of  $J \subset (a, b)$  simplifies to:

$$L_k \chi_J(y) = g_k \chi_{f_k(I_k \cap J)}(y). \quad (13)$$

We extend  $f_k$  to a map of point germs  $\widehat{u} = u^\epsilon \in [a_k^+, b_k^-]$  by setting  $\widehat{f}_k(u^\epsilon) = (u')^{\epsilon'}$ , where  $u' = f_k(u^\epsilon)$  and  $\epsilon' = s_k \epsilon$  (the direction of the point germ is reversed precisely

when  $f_k$  reverses orientation). By the results of the previous section, the action of  $L_k$  on an SBV-function is uniquely determined by its action upon a base step function. When  $a_k < \hat{u} < b_k$  the action of  $L_k$  upon  $\sigma_{\hat{u}}$  yields a simple step function with at most 3 jumps (see Figure 1) which may therefore be written as the sum of (at most) 3 base step functions. When  $\hat{u}$  is outside of the interval  $[a_k^+, b_k^-]$ , (at most) 2 such base step functions suffice. Taking into account the reversal of orientation when  $s_k = -1$  we may summarize as follows :

$$L_k \sigma_{\hat{u}} = \frac{1}{2} s_k g_k \times \begin{cases} +\sigma_{\hat{f}_k a_k^+} - \sigma_{\hat{f}_k b_k^-}, & a < \hat{u} < a_k \\ -\sigma_{\hat{f}_k a_k^+} - \sigma_{\hat{f}_k b_k^-} + 2\sigma_{\hat{f}_k \hat{u}} & a_k < \hat{u} < b_k \\ -\sigma_{\hat{f}_k a_k^+} + \sigma_{\hat{f}_k b_k^-}, & b_k < \hat{u} < b \end{cases} . \quad (14)$$

The term  $\sigma_{\hat{f}_k \hat{u}}$  only occurs when  $a_k < \hat{u} < b_k$  while both  $\sigma_{\hat{f}_k a_k^+}$  and  $\sigma_{\hat{f}_k b_k^-}$  always appear, but with a sign depending upon the position of  $\hat{u}$  relative to  $a_k$  and  $b_k$ , respectively. We may collect terms and write:

$$L_k \sigma_{\hat{u}} = s_k g_k \left[ \chi_{(a_k, b_k)}(\hat{u}) \sigma_{\hat{f}_k \hat{u}} - \sigma_{a_k}(\hat{u}) \sigma_{\hat{f}_k a_k^+} + \sigma_{b_k}(\hat{u}) \sigma_{\hat{f}_k b_k^-} \right] . \quad (15)$$

Here,  $\sigma_{a_k}(\hat{u}) = -\sigma_{\hat{u}}(a_k)$  takes the value  $+1/2$  when  $\hat{u} > a_k$  and  $-1/2$  when  $\hat{u} < a_k$  (and similarly for  $\sigma_{b_k}(\hat{u})$ ). The essential point is that the RHS is again a sum of elementary step functions. Thus, acting with a linear functional on both sides we obtain a representation for the dual of the Ruelle operator:

$$\widehat{L}_k \widehat{\ell}(\hat{u}) := \langle L'_k \ell, \sigma_{\hat{u}} \rangle = \langle \ell, L_k \sigma_{\hat{u}} \rangle \quad (16)$$

$$= s_k g_k \left[ \chi_{(a_k, b_k)}(\hat{u}) \widehat{\ell}(\hat{f}_k \hat{u}) - \sigma_{a_k}(\hat{u}) \widehat{\ell}(\hat{f}_k a_k^+) + \sigma_{b_k}(\hat{u}) \widehat{\ell}(\hat{f}_k b_k^-) \right] . \quad (17)$$

We extend this to any  $H \in \widehat{X}$  by setting

$$\widehat{L}_k H(\hat{u}) = s_k g_k \left[ \chi_{(a_k, b_k)}(\hat{u}) H(\hat{f}_k \hat{u}) - \sigma_{a_k}(\hat{u}) H(\hat{f}_k a_k^+) + \sigma_{b_k}(\hat{u}) H(\hat{f}_k b_k^-) \right] . \quad (18)$$

The constant function  $\mathbf{1}$  is in the kernel of  $\widehat{L}_k$ , since for every  $\hat{u} \in [a^+, b^-]$ :

$$\widehat{L}_k \mathbf{1}(\hat{u}) = s_k g_k \left[ \chi_{(a_k, b_k)}(\hat{u}) - \sigma_{a_k}(\hat{u}) + \sigma_{b_k}(\hat{u}) \right] \equiv 0. \quad (19)$$

One also has  $\widehat{L}_k H(a^+) = \frac{1}{2} s_k g_k \left( H(\hat{f}_k a_k^+) - H(\hat{f}_k b_k^-) \right) = -\widehat{L}_k H(b^-)$ . Consequently,  $\widehat{L}_k H(a^+) + \widehat{L}_k H(b^-) = 0$ , so  $\widehat{L}_k$  not only preserves  $\widehat{X}_0$  but maps  $\widehat{X}$  into  $\widehat{X}_0$ . In particular, any (non-zero) spectral property does not change by considering the extended operator. In order to simplify expressions, we introduce the following two operators acting upon  $H \in \widehat{X}$ :

$$S_k H(\hat{u}) = s_k g_k \chi_{(a_k, b_k)}(\hat{u}) H(\hat{f}_k \hat{u}), \quad (20)$$

$$P_k H(\hat{u}) = \sigma_{a_k}(\hat{u}) H(a_k^+) - \sigma_{b_k}(\hat{u}) H(b_k^-). \quad (21)$$

In terms of these, our dual representation may be written as

$$\widehat{L}_k = S_k - P_k S_k. \quad (22)$$

The weighted composition operator  $S_k$  is (just) a bounded linear weighted composition operator, but the image of each  $P_k$  is spanned by the elements  $\sigma_{a_k}$  and  $\sigma_{b_k}$  so  $P_k$  is of rank two. Our representation of the dual Ruelle operator is therefore a finite rank perturbation of  $S_k$ .

As the reader may verify, neither  $S_k$  nor  $P_k S_k$  need preserve  $\widehat{X}_0$  (although their difference does). It is, however, somewhat easier to work with  $S_k$  and  $P_k$  acting upon  $\widehat{X}$ , rather than making identifications to restrict their action to  $\widehat{X}_0$ .

### 3.1 A dynamical system

We consider here the situation when  $I_k = (a_k, b_k) = (c_k, c_{k+1})$  with  $a = c_0 < c_1 < \dots < c_{d+1} = b$ . The intervals are disjoint and their union fill out  $(a, b)$  apart from the cutting points  $c_1, \dots, c_d$ .

We associate as in the previous section a map, a weight and an operator to each interval. We write  $(I_k, f_k, g_k)$ ,  $k = 1, \dots, d$  for the collection of triples and denote by  $L = \sum_k L_k$  the Ruelle transfer operator associated to this collection.

By our choice of cutting points, we may extend the collection of  $\widehat{f}_k$ 's to a unique map  $\widehat{f}$  defined upon the entire set of point germs  $[a^+, b^-]$  (even if  $f$  need not extend to a continuous map on  $I$ !). Signs and weights also extend to functions of  $[a^+, b^-]$ . When  $\widehat{u} \in \widehat{I}_k = [c_k^+, c_{k+1}^-]$ , we write  $s(\widehat{u}) = s_k$ ,  $g(\widehat{u}) = g_k$  and we have the expression for the corresponding sum of composition operators:

$$SH(\widehat{u}) = \sum_k S_k H(\widehat{u}) = \sum_k s_k g_k \chi_{(a_k, b_k)}(\widehat{u}) H \circ \widehat{f}_k(\widehat{u}) = (sg)(\widehat{u}) H(\widehat{f} \widehat{u}).$$

Let us also write  $g^{(n)}(\widehat{u}) = g \circ \widehat{f}^{n-1}(\widehat{u}) \dots g(\widehat{u})$  (and similarly for  $s^{(n)}$ ) for the product of weights (or signs) along the orbit of  $\widehat{u}$ . Then  $S^n H(\widehat{u}) = (sg)^{(n)}(\widehat{u}) H(\widehat{f}^n \widehat{u})$  and the spectral radius of  $S$  is given by

$$\rho_\infty = \lim_n \|S^n\|_\infty^{1/n} = \lim_n \|g^{(n)}\|_\infty^{1/n}.$$

From the previous section we see that the dual of the Ruelle transfer operator,  $L' = \sum_k L'_k$ , is a finite rank perturbation of  $S$  so the essential spectral radius of  $L'$  (whence of  $L$ ) is not greater than  $\rho_\infty$ . If  $\rho(L) > \rho_\infty$  then  $L$  is quasi-compact ([DS, VIII.8]) and any spectral value  $\lambda$  with  $|\lambda| > \rho_\infty$  (we call such a value a peripheral spectral value) must be an isolated eigenvalue of  $L$  of finite algebraic multiplicity.

A function of the form  $S_k H$  has support in  $I_k$  so we have  $P_m S_k = 0$  whenever  $k \neq m$ . Whence,  $\sum_k P_k S_k = P S$ , where  $P = \sum P_k$ . Furthermore, we may recollect

terms to get:

$$PH(\widehat{u}) = \sum_{k=0}^d \left[ \sigma_{c_k}(\widehat{u})H(c_k^+) - \sigma_{c_{k+1}}(\widehat{u})H(c_{k+1}^-) \right] = \sigma_a(\widehat{u})\Delta_a H + \sum_{j=1}^d \sigma_{c_j}(\widehat{u})\Delta_{c_j} H ,$$

in which we have used  $\sigma_b(\widehat{u}) = -\sigma_a(\widehat{u}) = 1$  and defined

$$\Delta_{c_j} H = H(c_j^+) - H(c_j^-), \quad j \geq 1 , \quad (23)$$

$$\Delta_a H = H(a^+) + H(b^-) . \quad (24)$$

The image of  $P$  is  $d + 1$  dimensional and as  $\Delta_c \sigma_{c'} = \delta_{c,c'}$ ,  $P$  is also a projection. If  $|\lambda| > \rho_\infty$  then  $\lambda - S$  is invertible through a von Neumann series so  $\lambda - \widehat{L} = (1 + PS(\lambda - S)^{-1})(\lambda - S)$ . Equivalently, writing  $t = 1/\lambda$ :

$$1 - t\widehat{L} = G(t)(1 - tS), \quad (25)$$

where

$$G(t) = 1 + t PS(1 - tS)^{-1} = (1 - P) + P(1 - tS)^{-1}. \quad (26)$$

We conclude that  $\lambda - \widehat{L}$  is non-invertible (i.e.  $\lambda = 1/t$  is a spectral value) iff the operator  $G(t)$  is non-invertible.

Since  $(1 - P)G(t) = (1 - P)$  it suffices to look at the restriction of  $G(t)$  to the image of  $P$  which is spanned by  $\sigma_{c_0} = \sigma_a$  and  $\sigma_{c_1}, \dots, \sigma_{c_d}$ . Acting upon these  $d + 1$  functions we get the following  $(d + 1) \times (d + 1)$  matrix representation

$$G(t) \sigma_{c_k} = P(1 - tS)^{-1} \sigma_{c_k} = \sum_{j=0}^d \sigma_{c_j} M_{jk}(t)$$

where

$$M_{jk}(t) = \Delta_{c_j} \left( (1 - tS)^{-1} \sigma_{c_k} \right) = \Delta_{c_j} \Theta_{c_k}(\cdot, t) \quad (27)$$

and

$$\Theta_{c_k}(\widehat{x}, t) = \left( (1 - tS)^{-1} \sigma_{c_k} \right) (\widehat{x}) = \sum_{t \geq 0} t^n (sg)^n(\widehat{x}) \sigma_{c_k}(\widehat{f}^n \widehat{x}), \quad (28)$$

is equivalent to the so-called kneading coordinate of Milnor-Thurston. Our matrix  $M(t)$  is of size  $(\ell + 1) \times (\ell + 1)$  and is one version of the Milnor-Thurston kneading matrix. The original [MT] is,  $\ell \times (\ell + 1)$  dimensional) but the associated M-T determinant is identical to the determinant of  $M(t)$  as shown in [RT, Appendix B].

The value  $\lambda = 1/t$  is an eigenvalue iff  $d_{\text{MT}}(t) := \det M(t) = 0$ . Moreover, the order of the zero of the determinant equals the algebraic multiplicity of the eigenvalue. This last property is known as the Weinstein Aronszajn formula (while not difficult to show it is somewhat lengthier, so we refer to e.g. [Kato, Ch. IV.6] or [Ru, Lemma 2.8] for a proof). In summary we have shown:



**Theorem 3.1.** *Let  $\sigma_{\text{sp}}(L)$  be the spectrum of the Ruelle transfer operator acting upon  $X = \text{SBV}(a, b)$ . The peripheral spectrum,  $\{z \in \sigma_{\text{sp}}(L) : |z| > \rho_{\infty}\}$  consists of isolated eigenvalues of finite multiplicity only. Furthermore,  $\lambda$  is a peripheral eigenvalue iff  $t = 1/\lambda$  is a zero of the Milnor Thurston determinant  $d_{\text{MT}}(t)$  and the algebraic multiplicity of  $\lambda$  equals the order of the zero of  $t$ .*

**Remarks 3.2.** *In the above we have used a representation of a simple step function  $\phi$  in which the value at each point  $x \in (a, b)$  is specified independently of the limit-values at  $x^+$  and  $x^-$ . Other possibilities are to assign the value  $\frac{1}{2}(\phi(x^-) + \phi(x^+))$  at  $x$  or to declare two functions equivalent if they agree except at a countable set. The dual space of the completion in both of the latter cases may be identified with  $\{H \in \ell^{\infty}([a, b]) : H(a) + H(b) = 0\}$ . This is, however, inconvenient when the map is discontinuous and the forward orbits of  $c_j^+$  and  $c_j^-$  are distinct. With our approach the distinction is automatic as it is build into the space of point-germs.*

### 3.2 Positivity and leading zero

The most interesting situation is when all weights are non-negative and the Ruelle transfer operator is positive, i.e. preserves the cone of non-negative functions. We may use this to give a direct simple proof of the extremal properties of the zero as alluded to in the introduction. Let  $Z_n$  denote the collection of monotonicity intervals (also known as cylinder sets) of  $f^n$ . We define the partition function by

$$\Omega_n = \sum_{\alpha \in Z_n} g_{|\alpha}^{(n)} \quad (29)$$

The particular case  $g_k \equiv 1$  for every  $k$ , yields the topological entropy through  $h_{\text{top}} = \log \lim_n \Omega_n^{1/n}$ .

**Theorem 3.3.** *Suppose that all  $g_k \geq 0$ . Then we have*

$$\rho_1 := \lim_n \Omega_n^{1/n} = \lim_n \|L^n \mathbf{1}\|_{\infty}^{1/n} = \rho_{\text{sp}}(L). \quad (30)$$

*Moreover, if  $\rho_{\text{sp}}(L) > \rho_{\infty}$  then  $\rho_{\text{sp}}(L)$  is an isolated eigenvalue of  $L$  of finite multiplicity and  $t^* = 1/\rho_{\text{sp}}(L) > 0$  is the smallest zero (in absolute value) of the Milnor-Thurston determinant  $d_{\text{MT}}(t)$ .*

Proof: As  $L^n \mathbf{1}(y) \leq \Omega_n$  for all  $n$  and  $y$  we have  $\rho_{\infty} \leq \lim_n \|L^n \mathbf{1}\|^{1/n} \leq \rho_1$ .

Consider the set  $Z_n$  of (non-empty) monotonicity intervals for  $f^n$ . Any open interval  $\alpha \in Z_n$  may be written in terms of point germs as  $\alpha = \langle \hat{u}, \hat{v} \rangle$  where each directed end point is either a cutting point itself or it maps into a cutting point within at most  $n-1$  iterations. Let  $0 \leq k < n$  be the smallest such number for which  $\hat{f}^k(\hat{x}) = \hat{c}$  where  $\hat{c}$  is either  $c_j^-$  or  $c_j^+$  for some  $1 \leq j \leq d$ . Now, the partition sum is the sum of  $g_{|\alpha}^{(n)}$  over all

$\alpha \in Z_n$ . We get twice the value if we instead sum over all interval endpoints. Using that  $g^{(n)}(\hat{x}) = g^{(k)}(\hat{x})g^{(n-k)}(\hat{c}_j)$  and rewriting the sum in view of the previous remarks we get the identity (the boundary points  $a^+$  and  $b^-$  need a special treatment):

$$2\Omega_n = \sum_{\langle \hat{u}, \hat{v} \rangle \in Z_n} \sum_{\hat{x} \in \{\hat{u}, \hat{v}\}} g^{(n)}(\hat{x}) = g^{(n)}(a^+) + g^{(n)}(b^-) + \sum_{j=1}^d \sum_{k=0}^{n-1} L^k \mathbf{1}(c_j) \sum_{\hat{c}=c_j^\pm} g^{(n-k)}(\hat{c}). \quad (31)$$

We deduce the inequality  $2\Omega_n \leq 2\|g^{(n)}\|_\infty + d \sum_{k=0}^{n-1} \|L^k \mathbf{1}\|_\infty \|g^{(n-k)}\|_\infty$  and then:

$$\rho_1 = \lim_{n \rightarrow \infty} \Omega_n^{1/n} \leq \lim_{n \rightarrow \infty} \|L^n \mathbf{1}\|_\infty^{1/n} \leq \|L^n \mathbf{1}\|_X^{1/n} \leq \rho_{sp}(L).$$

If  $\rho_{sp}(L) = \rho_\infty$  we are through. So suppose that  $\rho_{sp}(L) > \rho_\infty$ . As any peripheral spectral value is an eigenvalue of finite multiplicity, there must be  $\lambda \in \mathbb{C}$  and a non-trivial  $\phi \in X$  with  $\lambda\phi = L\phi$  and  $|\lambda| = \rho_{sp}(L)$ . Taking absolute values we get by positivity of the operator

$$|\lambda| |\phi| = \left| \sum_k g_k \phi \circ f_k^{-1} \chi_{f_k I_k} \right| \leq \sum_k g_k |\phi| \circ f_k^{-1} \chi_{f_k I_k} = L|\phi|. \quad (32)$$

Similarly  $|\lambda|^n |\phi| \leq L^n |\phi| \leq L^n \mathbf{1} \|\phi\|_\infty$ , from which we deduce that

$$\rho_{sp}(L) = |\lambda| \leq \lim_n \|L^n \mathbf{1}\|^{1/n} \leq \rho_1 \leq \rho_{sp}(L).$$

For the last assertion, consider the above  $\phi$  and  $\lambda$ . When  $0 < t < 1/\rho_{sp}(L)$  we get  $(\mathbf{1} - tL)^{-1} |\phi| = \sum t^k L^k |\phi| \geq \sum t^k |\lambda|^k |\phi| = (1 - t \rho_{sp}(L))^{-1} |\phi|$  which diverges as  $t \rightarrow (1/\rho_{sp}(L))^-$ . This implies that  $(\rho_{sp}(L)\mathbf{1} - L)$  must be non-invertible and whence that  $\rho_{sp}(L)$  is an eigenvalue of  $L$ . Then  $t_* = 1/\rho_{sp}(L)$  is a zero of  $d_{MT}(t)$ , and the smallest such since there can be no spectral value larger than  $\rho_{sp}(L)$ .  $\square$

**Examples 3.4.** Consider the two triples of maps (with unit weights):

$I_1 = (0, \frac{1}{2}), f_1(x) = 2x, g_1 = 1$  and  $I_2 = (\frac{1}{2}, 1), f_2(x) = x - \frac{1}{2}, g_2 = 1$ .  $\hat{f}$  maps the four directed boundary points  $0^+, \frac{1}{2}^-, \frac{1}{2}^+$  and  $1^-$  to  $0^+, 1^-, 0^+$  and  $\frac{1}{2}^-$ , respectively.  $\Theta_{\frac{1}{2}}(\frac{1}{2}^+, t)$  is calculated by looking at the forward orbit of  $\frac{1}{2}^+$  relative to  $\frac{1}{2}$ . For example, as  $\frac{1}{2}^+ > \frac{1}{2}$  (which yields a plus sign) and  $\hat{f}(\frac{1}{2}^+) = 0^+ = \hat{f}(0^+) < \frac{1}{2}$  (so a minus sign) we get

$$\Theta_{\frac{1}{2}}(\frac{1}{2}^+, t) = \frac{1}{2} (1 - t - t^2 - t^3 - \dots) = \frac{1}{2} \frac{1 - 2t}{1 - t}. \quad (33)$$

Our  $2 \times 2$  kneading matrix, cf. (23), (24) and (27), then becomes

$$M(t) = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix} = \begin{pmatrix} \Theta_0(0^+, t) + \Theta_0(1^-, t) & \Theta_{\frac{1}{2}}(0^+, t) + \Theta_{\frac{1}{2}}(1^-, t) \\ \Theta_0(\frac{1}{2}^+, t) - \Theta_0(\frac{1}{2}^-, t) & \Theta_{\frac{1}{2}}(\frac{1}{2}^+, t) - \Theta_{\frac{1}{2}}(\frac{1}{2}^-, t) \end{pmatrix} \quad (34)$$

$$= \frac{1}{2} \begin{pmatrix} \frac{1}{1-t} + \frac{1}{1-t} & \frac{1}{1-t} + \frac{-1}{1+t} \\ \frac{1}{1-t} - \frac{1}{1-t} & \frac{1-2t}{1-t} - \frac{-1-2t}{1+t} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-t} & \frac{-t}{1-t^2} \\ 0 & \frac{1-t-t^2}{1-t^2} \end{pmatrix}. \quad (35)$$

The determinant is  $d_{\text{MT}}(t) = \frac{1-t-t^2}{(1-t)^2(1+t)}$  with one zero at  $t = 2/(1 + \sqrt{5}) = 1/\gamma$  (implying  $h_{\text{top}} = \log \gamma$ ) and no other zeroes in the unit disk. In the case of topological entropy it suffices in fact to consider the minor obtained from  $M(t)$  by erasing the first line and the first column (see e.g. [RT, Appendix B]).

The corresponding Ruelle determinant  $d_{\text{R}}(t)$  (or reciprocal zeta-function) is calculated from the sequence  $\#\text{Fix} \hat{f}^n$ ,  $n \geq 1$ , the number of fixed points of the  $n$ -th iterate of the map (see e.g. [BK]):

$$d_{\text{R}}(t) = \exp \left( - \sum_{n \geq 1} \frac{t^n}{n} \#\text{Fix} \hat{f}^n \right) = \exp \left( - \sum_{n \geq 1} \frac{t^n}{n} \text{tr} T^n \right) \quad (36)$$

$$= \det(\mathbf{1} - tT) = 1 - t - t^2, \quad (37)$$

where  $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  is the transition matrix corresponding to  $\widehat{f}\widehat{I}_1 = \widehat{I}_1 \cup \widehat{I}_2$  and  $\widehat{f}\widehat{I}_2 = \widehat{I}_1$ .

The zeros of this determinant are (see [BK]) the peripheral eigenvalues of  $L$  (counted with multiplicity). In accordance with our main theorem,  $d_{\text{R}}(T)/d_{\text{MT}}(t) = (1-t)^2(1+t)$  is analytic and without zeros in the unit disk. We note that the precise form of  $d_{\text{R}}(t)$  (but not the conclusion) depends somewhat upon the choice of how to treat periodic orbits on the boundaries of the intervals. In the above example, they may be omitted in the sum, in which case one recovers for  $d_{\text{R}}(t)$  the (reciprocal of the) Lefschetz zeta function which is identical to  $d_{\text{MT}}(t)$  as already shown in [MT].

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